C3 JUNE 10

1. (a) Show that

$$\frac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$$

(2)

(b) Hence find, for $-180° \leqslant \theta < 180°$, all the solutions of

$$\frac{2\sin 2\theta}{1+\cos 2\theta} = 1$$

Give your answers to 1 decimal place.

(3)

a) $\frac{2\sin\theta\cos\theta}{1+(2\cos^2\theta-1)} = \frac{\cancel{2}\sin\theta\cancel{\cos\theta}}{\cancel{2}\cos^2\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

qed

b) $2\left(\frac{\sin 2\theta}{1+\cos 2\theta}\right) = 1 \Rightarrow \tan\theta = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right)$

$\theta = \underline{26.6°}; \ \underline{-153.4°}$

2. A curve C has equation

$$y = \frac{3}{(5-3x)^2}, \quad x \neq \frac{5}{3}$$

$$y = 3(5-3x)^{-2}$$

The point P on C has x-coordinate 2. Find an equation of the normal to C at P in the form $ax + by + c = 0$, where a, b and c are integers.

(7)

$x = 2 \quad y = \frac{3}{(5-6)^2} = 3 \quad P(2,3)$

$\frac{dy}{dx} = -6(5-3x)^{-3} \times -3 = \frac{18}{(5-3x)^3}$

$x = 2 \Rightarrow m_t = \frac{18}{(5-6)^3} = -18 \Rightarrow m_n = \frac{1}{18}$

$y - 3 = \frac{1}{18}(x-2) \Rightarrow 18y - 54 = x - 2$

$x - 18y + 52 = 0.$

3.
$$f(x) = 4\operatorname{cosec} x - 4x + 1, \quad \text{where } x \text{ is in radians.}$$

(a) Show that there is a root α of $f(x) = 0$ in the interval $[1.2, 1.3]$.
(2)

(b) Show that the equation $f(x) = 0$ can be written in the form

$$x = \frac{1}{\sin x} + \frac{1}{4}$$
(2)

(c) Use the iterative formula

$$x_{n+1} = \frac{1}{\sin x_n} + \frac{1}{4}, \qquad x_0 = 1.25,$$

to calculate the values of x_1, x_2 and x_3, giving your answers to 4 decimal places.
(3)

(d) By considering the change of sign of $f(x)$ in a suitable interval, verify that $\alpha = 1.291$ correct to 3 decimal places.
(2)

$f(1\cdot2) = \dfrac{4}{\sin 1\cdot2} - 4(1\cdot2) + 1 = 0\cdot49 \quad f(1\cdot2) > 0$

$f(1\cdot3) = \dfrac{4}{\sin 1\cdot3} - 4(1\cdot3) + 1 = -0\cdot0487 \quad f(1\cdot3) < 0$

by sign change rule root lies between 1·2 and 1·3

b) $\dfrac{4}{\sin x} - 4x + 1 = 0 \Rightarrow 4x = \dfrac{4}{\sin x} + 1 \Rightarrow x = \dfrac{1}{\sin x} + \dfrac{1}{4}$

c) $x_1 = 1\cdot3038$; $x_2 = 1\cdot2867$; $x_3 = 1\cdot2917$

d) $f(1\cdot2905) = 0\cdot000446$ } $f(x)$ is continuous so
$f(1\cdot2915) = -0\cdot00475$ } by sign change rule

root lies between 1·2905 and 1·2915

$\Rightarrow \alpha = 1\cdot291$ (3dp)

4. The function f is defined by

$$f : x \mapsto |2x-5|, \quad x \in \mathbb{R}$$

(a) Sketch the graph with equation $y = f(x)$, showing the coordinates of the points where the graph cuts or meets the axes.

(2)

(b) Solve $f(x) = 15 + x$.

(3)

The function g is defined by

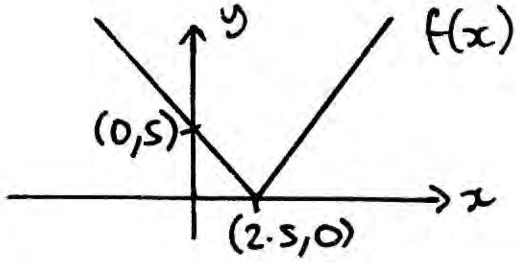
$$g : x \mapsto x^2 - 4x + 1, \quad x \in \mathbb{R}, \quad 0 \leqslant x \leqslant 5$$

(c) Find fg(2).

(2)

(d) Find the range of g.

(3)



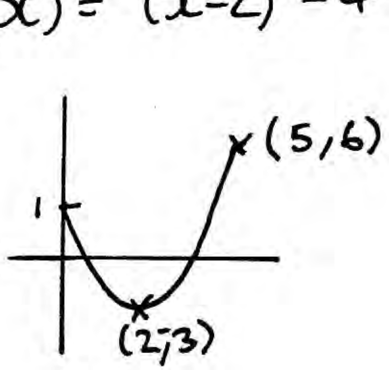
$|2x-5| = 15+x$ $\quad 2x-5=15+x$ $\quad -(2x-5)=15+x$

$\underline{x=20}$ $\quad -10=3x$

$x = \underline{\frac{-10}{3}}$

c) $fg(2) = f(2^2-4(2)+1) = f(-3) = |-6-5| = \underline{11}$

d) $g(x) = (x-2)^2-4+1 = (x-2)^2-3$



range

$-3 \leq y \leq 6$

5.

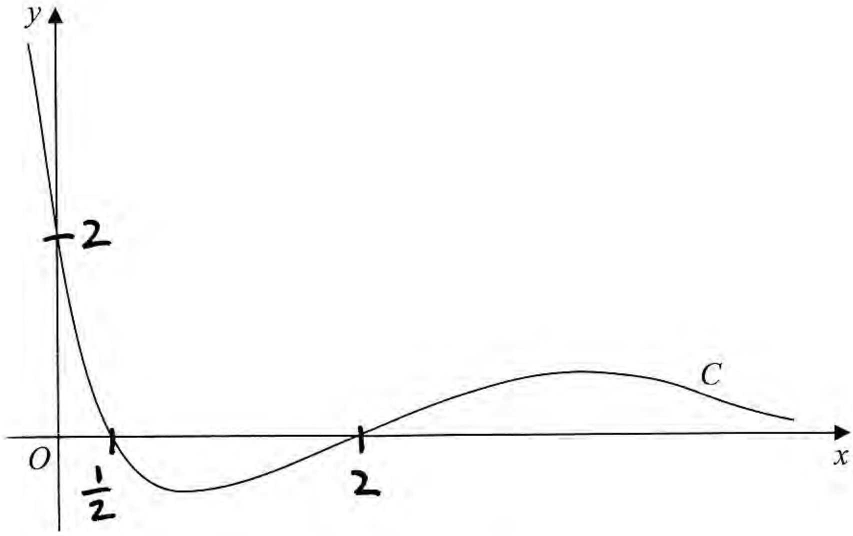


Figure 1

Figure 1 shows a sketch of the curve C with the equation $y = (2x^2 - 5x + 2)e^{-x}$.

(a) Find the coordinates of the point where C crosses the y-axis. $y=2$ **(1)**

(b) Show that C crosses the x-axis at $x = 2$ and find the x-coordinate of the other point where C crosses the x-axis.

(3)

(c) Find $\frac{dy}{dx}$.

(3)

(d) Hence find the exact coordinates of the turning points of C.

(5)

b) $e^{-x} \neq 0 \Rightarrow (2x^2-5x+2) = 0 = (2x-1)(x-2)$

$x=\frac{1}{2}$ $x=2$

c) $u=2x^2-5x+2$ $v=e^{-x}$

$u' = 4x-5$ $v'=-e^{-x}$

$\frac{dy}{dx} = (4x-5)e^{-x} - (2x^2-5x+2)e^{-x} = (9x-7-2x^2)e^{-x}$

d) at TP $\frac{dy}{dx}=0$ $e^{-x} \neq 0 \Rightarrow 9x-7-2x^2=0$

$2x^2+7-9x = (2x-7)(x-1) = 0$ $x=\frac{7}{2}$ $x=1$

$x=1 \quad y=(2-5+2)e^{-1} = -e^{-1} \qquad \left(1, -\frac{1}{e}\right)$

$x=\frac{7}{2} \quad y=\left(2\left(\frac{7}{2}\right)^2-5\left(\frac{7}{2}\right)+2\right)e^{-\frac{7}{2}} = 9e^{-\frac{7}{2}} \qquad \left(\frac{7}{2}, 9e^{-\frac{7}{2}}\right)$

6.

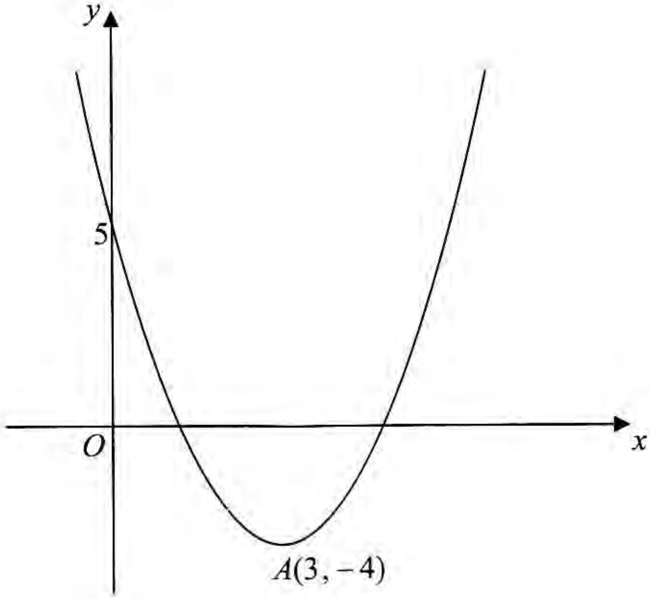


Figure 2

Figure 2 shows a sketch of the curve with the equation $y = f(x)$, $x \in \mathbb{R}$.
The curve has a turning point at $A(3, -4)$ and also passes through the point $(0, 5)$.

(a) Write down the coordinates of the point to which A is transformed on the curve with equation

(i) $y = |f(x)|$, (3,4)

(ii) $y = 2f(\frac{1}{2}x)$. (6,-8)

(4)

(b) Sketch the curve with equation

$$y = f(|x|)$$

On your sketch show the coordinates of all turning points and the coordinates of the point at which the curve cuts the y-axis.

(3)

The curve with equation $y = f(x)$ is a translation of the curve with equation $y = x^2$.

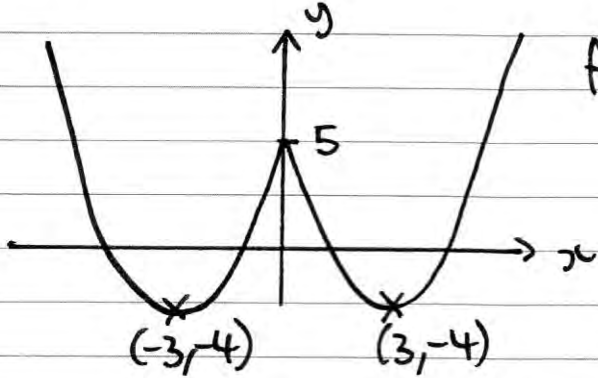
(c) Find $f(x)$.

(2)

(d) Explain why the function f does not have an inverse.

(1)

b)



c) translate 3 horizontally; 4 vertically down

$f(x-3)-4 = (x-3)^2-4$

d) $f(x)$ is <u>NOT</u> one to one; so it can not have an inverse

(it could have an inverse if the domain is restricted $x \geq 3$ or $x \leq -3$)

7. (a) Express $2\sin\theta - 1.5\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Give the value of α to 4 decimal places.

(3)

(b) (i) Find the maximum value of $2\sin\theta - 1.5\cos\theta$.

(ii) Find the value of θ, for $0 \leqslant \theta < \pi$, at which this maximum occurs.

(3)

Tom models the height of sea water, H metres, on a particular day by the equation

$$H = 6 + 2\sin\left(\frac{4\pi t}{25}\right) - 1.5\cos\left(\frac{4\pi t}{25}\right), \quad 0 \leqslant t < 12,$$

where t hours is the number of hours after midday.

(c) Calculate the maximum value of H predicted by this model and the value of t, to 2 decimal places, when this maximum occurs.

(3)

(d) Calculate, to the nearest minute, the times when the height of sea water is predicted, by this model, to be 7 metres.

(6)

a) $\frac{R\sin\alpha = 1.5}{R\cos\alpha = 2} \Rightarrow \tan\alpha = \frac{3}{4} \Rightarrow \alpha = \tan^{-1}\left(\frac{3}{4}\right)$

$\alpha = 0.6435^c$

$R^2 = 1.5^2 + 2^2 \Rightarrow R = \sqrt{\frac{25}{4}} = \frac{5}{2}$ $\quad \frac{5}{2}\sin(\theta - 0.6435)$

b) i) $\max = \frac{5}{2}$ ii) $(\theta - 0.6435) = \frac{\pi}{2} \Rightarrow \theta = 2.214^c$

c) $\max = 6 + 2.5 = 8.5\text{m}$ $\quad \frac{4\pi t}{25} = 2.214$

$t = 4.41$ hrs

d) $7 = 6 + \frac{5}{2}\sin(\theta - 0.6435) \Rightarrow \sin\left(\frac{4\pi t}{25} - 0.6435\right) = \frac{2}{5}$

$\frac{4\pi t}{25} - 0.6435 = 0.4115\ldots;\ 2.73\ldots$

$t = 2.0988;\ 6.7115\ldots$ $\quad t = 2\text{hr } 6\text{min};\ 6\text{hr } 43\text{min}$

8. (a) Simplify fully

$$\frac{2x^2 + 9x - 5}{x^2 + 2x - 15}$$

(3)

Given that

$$\ln(2x^2 + 9x - 5) = 1 + \ln(x^2 + 2x - 15), \quad x \neq -5,$$

(b) find x in terms of e.

(4)

a) $\dfrac{(2x-1)\cancel{(x+5)}}{\cancel{(x+5)}(x-3)} = \dfrac{2x-1}{x-3}$

b) $\ln(2x^2+9x-5) - \ln(x^2+2x-15) = 1$

$\ln\left(\dfrac{2x^2+9x-5}{x^2+2x-15}\right) = 1 \Rightarrow \ln\left(\dfrac{2x-1}{x-3}\right) = 1$

$\dfrac{2x-1}{x-3} = e \Rightarrow 2x-1 = ex-3e$

$\Rightarrow x(2-e) = 1-3e \Rightarrow x = \dfrac{1-3e}{2-e}$